Published by Institute of Physics Publishing for SISSA/ISAS

Received: *February 19, 2004*
Revised: *May 13, 2004*
Accepted: *June 28, 2004*

Scalar hair of global defect and black brane world

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Abstract: We consider a complex scalar field in $(p+3)$-dimensional bulk with a negative cosmological constant and study global vortices in two extra-dimensions. We reexamine carefully the coupled scalar and Einstein equations, and show that the boundary value of scalar amplitude at infinity of the extra-dimensions should be smaller than vacuum expectation value. The brane world has a cigar-like geometry with an exponentially decaying warp factor and a flat thick p-brane is embedded. Since a coordinate transformation identifies the obtained brane world as a black p-brane world bounded by a horizon, this strange boundary condition of the scalar amplitude is understood as existence of a short scalar hair.

Keywords: Solitons Monopoles and Instantons, Extra Large Dimensions.

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1. Introduction

In the recent developments, two main streams of brane world scenario have been proposed: one with large extra-dimensions by Arkani-Hamed-Dimopoulos-Dvali (ADD) [1], and the other with a warp geometry by Randall-Sundrum (RS) [2, 3].

In ref. [3], RS considered a flat 3-brane in five-dimensional anti-de Sitter (AdS) bulk spacetime with one extra-dimension. Newtonian gravity was reproduced on the 3-brane, in its sufficiently low energy limit. Subsequently numerous related topics have been explored, and, among them, two natural extensions have been the brane world with two extra-dimensions [4] and construction of thick brane world [5]–[19].

In this paper, we revisit the brane world of warp geometry composed of a flat thick p-brane and two extra-dimensions with rotational symmetry. Before considering bulk fields to make the brane thick, we examine Einstein equations by assuming only a negative bulk cosmological constant, and obtain general static vacuum solutions under the ansatz of a metric with warp factor. Those solutions are characterized mainly by an arbitrary integration constant, and the geometry with an exponentially decaying warp factor is uniquely given only when a particular value of that parameter is chosen. The obtained unique brane world from a vacuum solution is identified with that of AdS branch in ref. [7, 18]. It is also confirmed that any black p-brane structure cannot be formed with a horizon at a finite radius.

Suppose that we have a complex scalar field with global U(1) symmetry in the bulk. Then global vortices, thick global p-branes in our context, may be formed due to the topological winding between the broken vacua and spatial infinity of the two extra-dimensions. Since these global defects are gravitating without cosmological constant, a physical singularity is unavoidable [20] and does not disappear for the cases of nonzero p [6]. Once a negative cosmological constant is added, then a logarithmically divergent energy becomes not so harmful because of the negative vacuum energy proportional to its spatial volume.

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The corresponding spacetime is free from the physical singularity but can support a charged Bañados-Teitelboim-Zanelli (BTZ) black hole structure [21]. Since the brane world with a warp geometry along this line means an extension from 0-brane to general flat p-brane, appearance of both a singularity-free brane world with an exponentially decaying warp factor [7]–[9] and [18] and structure of a black brane world [11, 16] is naturally understood. A noteworthy observation is the fact that the obtained brane world in terms of the warp metric coincides exactly with the interior of the black brane world bounded by the degenerated horizon. Since our brane world is a static patch, an intriguing question arises about the existence of a short scalar hair. Specifically, the question is whether or not the amplitude of a bulk complex scalar field can have the vacuum expectation value as its boundary condition at spatial infinity of the extra-dimensions. By analyzing the coupled field equations, we will show that the answer is negative and then the scalar amplitude at spatial infinity has a smaller value than the vacuum expectation value. In the context of black p-brane world, this strange boundary condition of the scalar amplitude is understood as existence of a short scalar hair at the degenerated horizon. In order to have vanishing scalar hair, either the limit of vanishing scalar potential or that of vanishing cosmological constant should be taken, but both of them are unwanted limits. We will also explain the existence of an extremal black p-brane through a Hodge duality between derivative term of the scalar phase and a dual five-form field strength tensor.

In section 2 we consider a bulk composed of a flat p-brane and rotationally symmetric two extra-dimensions, and obtain all the static AdS solutions under the metric with a warp factor. In section 3 a bulk complex scalar field is assumed and show that the boundary value of the scalar field for a global defect cannot arrive at the vacuum expectation value at infinity of the extra-dimensions. In section 4 we identify the boundary condition as the existence of a short scalar hair around the degenerated horizon of an extremal black brane world through a coordinate transformation. We conclude in section 5 with a few comments and discussions.

2. Vacuum solutions and warp geometry

In this section, we revisit the static warp geometry of a flat p-brane configuration in D-dimensional bulk spacetime. The extra-dimensions are two and no bulk matter is assumed except for the gravity with a negative bulk cosmological constant. We will obtain all the static vacuum solutions under the metric with warp factor, which are characterized by one integration constant, and will show that an exponentially decaying warp factor is obtained only when this parameter is chosen by a specific value.

We begin with the Einstein-Hilbert action in $(D = p+3)$-dimensions with a nonpositive bulk cosmological constant Λ

$$S_{\rm EH} = \int d^D x \sqrt{-g_D} \left[-\frac{M_*^{p+1}}{16\pi} (R + 2\Lambda) \right], \tag{2.1}$$

where M_* is the fundamental scale of higher dimensional gravity. The following ansatz for the metric is adopted for description of the warp geometry with convenience

$$\begin{aligned} ds^2 &= g_{AB}dx^A dx^B \\ &= e^{2A(r)}dx^\mu dx_\mu - dr^2 - C^2(r)d\theta^2\,, \end{aligned} \tag{2.2}$$

where indices, $A, B, \ldots$, denote those of D-dimensional bulk and μ stands for spacetime coordinates of the flat p-brane.

Einstein equations from the action (2.1) are simplified as

$$-\frac{C''}{C} - pA'' + A'\frac{C'}{C} = 0\,, \tag{2.3}$$

$$A'' + A'\frac{C'}{C} + (p+1)A'^2 = \frac{2|\Lambda|}{p+1}\,, \tag{2.4}$$

$$A'' - \frac{C''}{C} - pA'\frac{C'}{C} + (p+1)A'^2 = 0\,, \tag{2.5}$$

and they reduce to two first-order equations

$$\left(A' - \frac{C'}{C}\right)Ce^{(p+1)A} = \kappa\,, \tag{2.6}$$

$$\frac{p}{2}A'^2 + A'\frac{C'}{C} = -\frac{\Lambda}{p+1}\,, \tag{2.7}$$

where κ is an integration constant. In the remaining part of this section, let us examine the Einstein equations (2.6)–(2.7) and show that the unique solution of $\kappa = 0$ among a set of infinite AdS solutions can depict RS geometry with an exponentially decreasing warp factor.

When κ is zero, the first equation (2.6) gives $C = \alpha e^A$. Substituting it into the second equation (2.7), we have a set of exact solutions which are all the static solutions of vanishing κ

$$A_\pm(r) = \pm\frac{2\omega}{p+2}\,r + \beta_\pm\,, \tag{2.8}$$

$$C_\pm(r) = \alpha_\pm e^{A_\pm(r)} = \alpha_\pm e^{\pm\frac{2\omega}{p+2}r+\beta_\pm}\,, \tag{2.9}$$

where $\alpha_\pm$ and $\beta_\pm$ are integration constants and

$$\omega = \sqrt{\frac{(p+2)|\Lambda|}{2(p+1)}}\,. \tag{2.10}$$

Note that $\beta_\pm$ can be fixed to be zero with the aid of reparametrization of the coordinates x^μ. The solution (A_-, C_-) describes an exponentially decaying warp factor in front of both spacetime metric of the p-brane and angle variable θ of the extra-dimensions. Therefore, a neck of the two extra-dimensions becomes thin as radius r increases. It is a p-dimensional candidate of brane world of the RS type [7].

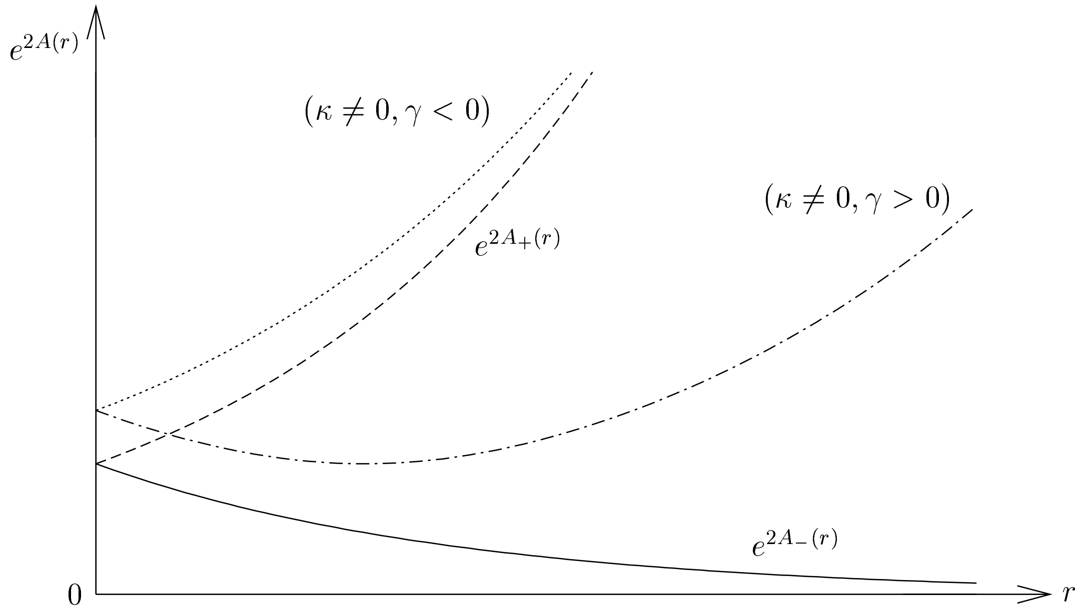

Figure 1: Four representative shapes of the metric function $e^{2A(r)}$: solid line for e^{2A_-}, dashed line for e^{2A_+}, and both dotted ($\gamma < 0$) and dashed-dotted ($\gamma > 0$) lines for the $\kappa \neq 0$ solutions.

When $\kappa \neq 0$, general solutions of the Einstein equations (3.5)–(3.7) are also obtained as

$$A(r) = \frac{2}{p+2} \ln\left[\cosh\left(\omega\, r - \gamma\right)\right] + \delta, \tag{2.11}$$
$$C(r) = -e^{\delta}\frac{\kappa}{\omega}\left[\cosh\left(\omega\, r - \gamma\right)\right]^{-\frac{p}{p+2}} \sinh\left(\omega\, r - \gamma\right), \tag{2.12}$$

where γ and δ are integration constants. Again, δ can be fixed to be zero with the help of reparametrization of the coordinates x^{μ}. A noteworthy property of the solutions is that the $\kappa \neq 0$ solutions are disconnected from the $\kappa = 0$ solutions, i.e., $\kappa \to 0$ limit of the $\kappa \neq 0$ solutions (2.11)–(2.12) do not coincide with the $\kappa = 0$ solutions (2.8)–(2.9). If we rewrite the solutions (2.11)–(2.12) by using exponential functions;

$$A(r) = \frac{2}{p+2} \ln(\gamma_+ e^{\omega r} + \gamma_- e^{-\omega r}), \tag{2.13}$$
$$C(r) = -\frac{\kappa}{\omega}(\gamma_+ e^{\omega r} + \gamma_- e^{-\omega r})^{-\frac{2}{p+2}}(\gamma_+ e^{\omega r} - \gamma_- e^{-\omega r}), \tag{2.14}$$

where $\gamma_{\pm} \equiv e^{\mp\gamma + \frac{p+2}{2}\delta - \ln 2}$. One may think that a solution with an exponentially decaying warp factor is obtained by taking the limit $\gamma_+ = 0$. However, when γ_+ is taken to be zero, the definition of γ_+ asks either γ or $-\delta$ to diverge to positive infinity so that γ_- becomes divergent or vanishes, respectively. Finally we only have either an unwanted solution $(\gamma_+ = 0, \gamma_- = \infty)$ or a trivial solution $(\gamma_+ = 0, \gamma_- = 0)$. The disconnectedness of the solutions is easily understood by appearance of a formal symmetry property for vanishing κ, that the first-order equations (2.6)–(2.7) possess a formal parity, $r \to -r$ as far as A' and $(\ln C)'$ share the same parity. As shown in figure 1, the $\kappa \neq 0$ solution has minimum value $e^{2A(r_{\min})} = e^{2\delta}(\cosh\gamma)^{2/(p+2)}$ at $r_{\min} = 0$ when γ is negative (see the dotted line in

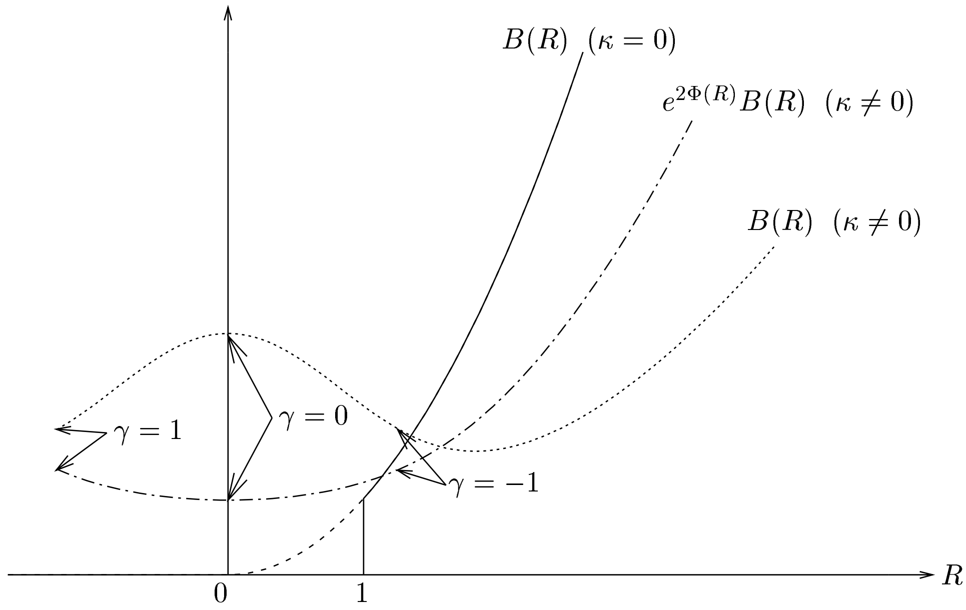

Figure 2: Plot of the various metric functions as functions of the new radial coordinate R: B (solid and dashed line) for the $\kappa = 0$ solutions, and $e^{2\Phi}B$ (dotted line) and B (dashed-dotted line) for the $\kappa \neq 0$ solutions. The points indicated by arrows on the curves stand for minimum values of R for given values of γ.

figure 1), but it has a minimum $e^{2A(r_{\min})} = e^{2\delta}$ at $r_{\min} = \gamma/\omega$ (see the dashed-dotted line in figure 1), when γ is positive. For sufficiently large r, both metric functions $e^{A(r)}$ and $C(r)$ asymptote exponentially growing solutions irrespective of value of any nonvanishing κ.

In conclusion, one of the $\kappa = 0$ solutions, $A_-(r)$ and $C_-(r)$ in eqs. (2.8) and (2.9) in ref. [7] (see the solid line in figure 1), is the unique static vacuum solution to describe RS type brane world with an exponentially decaying warp factor. Since geometry of our interest is given by a fine tuned solution but all the other static vacuum solutions have totally different asymptotic behavior, almost all the new solutions are likely to be connected with exponentially growing solutions.

A familiar coordinate for AdS spacetime is Poincaré coordinate

$$ds^2 = e^{2\Phi(R)}B(R)g_{\mu\nu}dx^\mu dx^\nu - \frac{dR^2}{B(R)} - R^2 d\theta^2 \,. \tag{2.15}$$

Throughout a coordinate transformation $R = 2\omega C(r)/(p+2)$ with the help of reparametrization of the p-brane coordinates $x^\mu/\beta_\pm \to x^\mu$ $(\mu = 0, 1, \ldots, p)$, the vacuum solutions of the warp factor $(A_\pm, C_\pm)$ in eqs. (2.8)–(2.9) are transformed to a well-known form of the metric such as $\Phi(R) = 0$ and $B(R) = R^2$. Since the $(+)$-solution corresponds to a patch of range $1 \leq R \leq \infty$ (see solid line in figure 2) and the $(-)$-solution to that of range $0 \leq R \leq 1$ (see dashed line in figure 2) under the coordinate transformation, spatial infinity $(r = \infty)$ of the $(-)$-solution is translated into a horizon at the origin $(R = 0)$. For a 0-brane case, it is nothing but the horizon of a BTZ black hole at its zero mass limit.

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Let us try a coordinate transformation from the warp coordinate (2.2) to the Poincaré coordinate (2.15) for the general solutions (2.11) and (2.12). Comparison between the two metrics (2.2) and (2.15) provides

$$R = \frac{2\omega}{p+2}C(r) \longrightarrow \left(e^{-\delta}\frac{p+2}{2\kappa}R\right)^2 = x^{-\frac{2p}{p+2}}(x^2-1)\,, \tag{2.16}$$

$$B(R) = \left(\frac{dC}{dr}\right)^2 = \frac{x^2}{x^2-1}\left(1+\frac{p}{2x^2}\right)^2 R^2\,, \tag{2.17}$$

$$\Phi(R) = A(r) - \ln\frac{p+2}{2\omega}\sqrt{B(R)} = \ln\left(e^{\delta}\frac{2\omega}{p+2}\frac{x^{\frac{p+4}{p+2}}\sqrt{x^2-1}}{x^2+p/2}\frac{1}{R}\right), \tag{2.18}$$

where $x = \cosh(\omega r - \gamma)$. If we rewrite eq. (2.16) in terms of a new variable $y \equiv x^{\frac{2}{p+2}}$, then we arrive at an algebraic equation

$$y^{p+2} - \left(e^{-\delta}\frac{p+2}{2\kappa}R\right)^2 y^p - 1 = 0\,. \tag{2.19}$$

When $p = 3$, it is obvious that any analytic form of a particular solution has not been reported from the above fifth-order algebraic equation except for the trivial case of $R = 0$. Therefore, the explicit coordinate transformation to the Poincaré coordinate cannot be obtained for the general $\kappa \neq 0$ vacuum solution (2.11) and (2.12). It may imply impossibility to solve the general solutions including eqs. (2.11) and (2.12) as a patch in the Poincaré coordinates (2.15) though we obtained those exact solutions in the warp coordinates (2.2). A rescaling of the spacetime variables of the p-brane as $2\omega e^{\delta}x^{\mu}/(p+2) \rightarrow \tilde{x}^{\mu}$ leads to

$$ds^2 = x^{\frac{4}{p+2}}\eta_{\mu\nu}d\tilde{x}^{\mu}d\tilde{x}^{\nu} - \frac{x^2(x^2-1)}{(x^2+p/2)^2}\frac{dR^2}{R^2} - R^2 d\theta^2\,. \tag{2.20}$$

Radial coordinate $R(r)$ is a monotonically-increasing function of r with the range

$$e^{\delta}\frac{2\kappa}{p+1}(\cosh\gamma)^{-\frac{p}{p+2}}\sinh\gamma \leq R \leq \infty\,, \tag{2.21}$$

where the minimum value of R is negative, zero, or positive according to that of γ. Since $1 \leq \cosh\gamma \leq x \leq \infty$, $\Phi(R)$ is finite and $B(R)$ is positive for every R, which means that no horizon structure can be realized from all the $\kappa \neq 0$ vacuum solutions as shown in figure 2.

3. Global defect with fine tuning

We showed that the brane world geometry with an exponentially decaying warp factor could only be obtained in $(p+3)$-dimensions as a static vacuum solution of a specific integration constant (2.8)$\sim$(2.9), and all the other solutions are those with an exponentially growing warp factor (2.11)$\sim$(2.12). In this section we will examine smooth global U(1) vortices and address a question whether or not such defect configurations form a new brane world with an exponentially decaying warp factor at its asymptote.

A gravitating global defect in two extra-dimensions is a global vortex solution of a field theory coupled to gravity with a spontaneously broken continuous global symmetry. A prototypical model has U(1) symmetry and a 'Mexican hat' potential:

$$S = \int d^D x \sqrt{-g_D} \left[-\frac{M_*^{p+1}}{16\pi}(R + 2\Lambda) + \frac{1}{2}\nabla_A \bar{\phi} \nabla^A \phi - \frac{\lambda}{4}(\bar{\phi}\phi - v^2)^2 \right]. \tag{3.1}$$

Here we consider a static global vortex in two spatial extra-dimensions, and its transverse space has p-dimensions - p-brane configurations in the spacetime of dimensions $D = p + 3$. We shall call this p-dimensional defect as *global p-brane* in what follows.

When n vortices in two extra-dimensions are superimposed at the origin, an ansatz for the global p-brane is

$$\phi = v f(r) e^{in\theta}. \tag{3.2}$$

Note that both the complex scalar field ϕ and its vacuum expectation value v has a canonical mass dimension of $(p+1)/2$. To be a nonsingular configuration, the scalar amplitude f should vanish at the origin, i.e., $f(0) = 0$ is a boundary condition.

In terms of dimensionless variables and quantities in the metric (2.2) and the action (3.1)

$$\sqrt{\lambda} v x^\mu \to x^\mu \ (\sqrt{\lambda} v r \to r), \qquad \sqrt{\lambda} v C \to C, \qquad \frac{\Lambda}{\lambda v^2} \to \Lambda, \qquad G_D = \frac{v^2}{M_*^{p+1}}, \tag{3.3}$$

scalar field equation is

$$f'' + \left[\frac{C'}{C} + (p+1)A' \right] f' - \left[\frac{n^2}{C^2} + (f^2 - 1) \right] f = 0. \tag{3.4}$$

Einstein equations are simplified as

$$-\frac{C''}{C} - pA'' + A'\frac{C'}{C} = 8\pi G_D f'^2, \tag{3.5}$$

$$A'' + A'\frac{C'}{C} + (p+1)A'^2 = \frac{2}{p+1}\left[|\Lambda| - 2\pi G_D (f^2 - 1)^2 \right], \tag{3.6}$$

$$A'' - \frac{C''}{C} - pA'\frac{C'}{C} + (p+1)A'^2 = 8\pi G_D \frac{n^2}{C^2} f^2. \tag{3.7}$$

One may notice that no explicit metric dependence on the right-hand side of both equations (3.5)–(3.6). In eq. (3.6), the effective cosmological constant $\lambda v^2[-|\Lambda| + 2\pi G_D(f^2-1)^2]$ appears instead of the cosmological constant $\lambda v^2 \Lambda$.

First of all, let us investigate the core region of global vortex, where the scalar amplitude $f(r)$ grows as the radius increases. Up to now we find no exact solution from the coupled equations of motion (3.4)–(3.7), we employ two approaches to obtain an approximate solution near the origin: One is to use power series solution of the exact equations near the origin and the other is to solve approximated equations exactly. Then we will compare both results. Since the scalar amplitude of a global vortex lives near the symmetric vacuum at its core, i.e., $f \approx 0$, forms of the approximated equations from eqs. (3.4)–(3.7)

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become the same as those of the vacuum (2.6) and (2.7) except for the change of an effective cosmological constant $2\pi G_D + \Lambda$ instead of Λ, which can have any signature. Therefore, we can use the exact solutions, eqs. (2.8)–(2.12): One with vanishing κ is

$$A_{\pm}(r) = \pm \frac{2\tilde{\omega}}{p+2}\, r + \tilde{\beta}_{\pm}\,, \tag{3.8}$$

$$C_{\pm}(r) = \tilde{\alpha}_{\pm} e^{A_{\pm}(r)} = \tilde{\alpha}_{\pm} e^{\pm\frac{2\tilde{\omega}}{p+2} r + \tilde{\beta}_{\pm}}\,, \tag{3.9}$$

where $\tilde{\alpha}_{\pm}$ and $\tilde{\beta}_{\pm}$ are integration constants and

$$\tilde{\omega} = \sqrt{\frac{(p+2)(2\pi G_D + \Lambda)}{2(p+1)}}\,. \tag{3.10}$$

Here $\tilde{\beta}_{\pm}$ can also set to be zero by a reparametrization of the p-brane spacetime coordinates x^{μ}. The other is general solutions of the nonvanishing κ

$$A(r) = \frac{2}{p+2} \ln\left[\cos\left(\tilde{\omega} r - \tilde{\gamma}\right)\right] + \tilde{\delta} \tag{3.11}$$

$$\approx \left(\frac{2}{p+2}\ln\cos\tilde{\gamma} + \tilde{\delta}\right) + \frac{2\tilde{\omega}}{p+2}(\tan\tilde{\gamma})r - \frac{\tilde{\omega}^2}{p+2}(1 - 2\tan^2\tilde{\gamma})r^2 + \cdots\,, \tag{3.12}$$

$$C(r) = -e^{-(p+1)\tilde{\delta}}\frac{\kappa}{\tilde{\omega}}\left[\cos\left(\tilde{\omega} r - \tilde{\gamma}\right)\right]^{-\frac{p}{p+2}} \sin\left(\tilde{\omega} r - \tilde{\gamma}\right) \tag{3.13}$$

$$\approx e^{-(p+1)\tilde{\delta}}\frac{\kappa}{\tilde{\omega}}\sin\tilde{\gamma}(\cos\tilde{\gamma})^{-\frac{p}{p+2}}\Bigg[1 - \left(\frac{p}{p+2}\tan\tilde{\gamma} - \cot\tilde{\gamma}\right)\tilde{\omega} r + {} + (p-1)\left(\frac{1}{p+2} + \frac{p}{2}\tan^2\tilde{\gamma}\right)\tilde{\omega}^2 r^2\Bigg]\,, \tag{3.14}$$

where $\tilde{\gamma}$ and $\tilde{\delta}$ are integration constants of which $\tilde{\delta}$ can again be fixed to be zero by a reparametrization of the p-brane spacetime coordinates x^{μ}. When the effective cosmological constant $2\pi G_D + \Lambda$ is positive, exact solutions of the approximated equations (3.8)–(3.14) are expressed by trigonometric functions instead of hyperbolic functions.

On the other hand, if we take into account the scalar field f, then series solutions near the origin are given from eqs. (3.4)–(3.7)

$$f(r) \approx f_0 r^{n/c_0}\,, \tag{3.15}$$

$$A(r) \approx a_0 - \frac{\tilde{\omega}^2}{p+2} r^2\,, \tag{3.16}$$

$$C(r) \approx c_0 r\left\{1 + \frac{1}{6}\left[\frac{2(p-1)\tilde{\omega}^2}{p+2} - f_0^2 \delta_{\frac{n}{c_0}1}\right] r^2\right\}\,, \tag{3.17}$$

where f_0, a_0, c_0 are undetermined constants determined by proper behavior at the opposite boundary. c_0 should be restricted to make n/c_0 a natural number for regularity and a_0 can be absorbed by a reparametrization of the p-brane spacetime coordinates x^{μ}. Comparison of eq. (3.16) with eq. (3.8) leads obviously to the conclusion that the $\kappa = 0$ solution is not consistent with any possible nonsingular vortex configuration. On the other hand, expansion of the $\kappa \neq 0$ solution (3.14) can be consistent with eq. (3.16) if $\tilde{\gamma}$ is zero, $\tilde{\delta}$ is identified with a_0, and $\kappa = -\frac{\tilde{\omega}}{(p+1)a_0}\ln(c_0/\tilde{\omega})$. Even if we take into account possible effect of the scalar amplitude f and vorticity n outside the vortex core, it seems unlikely to find a configuration of global vortex of which asymptotic geometry is smoothly connected to the brane world of an exponentially decaying warp factor, $A_-(r)$ and $C_-(r)$ in eqs. (2.8)–(2.9), given as an AdS vacuum solution. According to the above argument, even if we obtain a geometry of an exponentially decaying warp factor at asymptotes with the global vortices, it is irrelevant with that from the vacuum solution (2.8)–(2.9). We will show that it is indeed the case.

Once any brane world is assumed to be formed from a bulk global U(1) defect solution of a complex scalar theory, natural boundary condition for the scalar amplitude is to reach vacuum expectation value at spatial infinity, $f(\infty) = 1$, without doubt [7]–[9] and [11, 16, 18]. From here on, let us read possible sets of boundary conditions for the metric functions $A(r)$ and $C(r)$ at spatial infinity when $\lim_{r\to\infty} f(r) = 1$, and show that $f(\infty) = 1$ cannot be taken in order to obtain a new brane world geometry with an exponentially decaying warp factor formed by a global p-brane.

We consider two cases categorized by asking a criterion that whether or not the topological term $n^2 f/C^2$ in the scalar equation (3.4) is involved in determining the boundary conditions of the metric functions $A(r)$ and $C(r)$. First, if it is not the case, then the negative cosmological term dominates and the boundary conditions for $A(r)$ and $C(r)$ at spatial infinity reduce obviously to those of the pure broken vacuum so that asymptotic solutions should be consistent with either geometry of the exponentially decaying warp factor, $A_-(r)$ and $C_-(r)$ in eqs. (2.8) and (2.9), or a hyperboloid, $A_+(r)$ and $C_+(r)$ in eqs. (2.8) and (2.9) and $A(r)$ and $C(r)$ in eqs. (2.11) and (2.12). Second, when the topological term proportional to n^2 contributes to determination of the boundary conditions of the metric functions, which is specifically $T^t_t = T^{z_i}_{z_i} = T^r_r = -T^\theta_\theta = n^2/2C^2$ at spatial infinity. For the boundary configuration of the scalar field, $f(\infty) = 0$ and $n \neq 0$, we obtain resultant Einstein equations from eqs. (3.6) and (3.7):

$$A'' + A'\frac{C'}{C} + (p+1)A'^2 = -\frac{2|\Lambda|}{p+1}, \tag{3.18}$$

$$A'' - \frac{C''}{C} - pA'\frac{C'}{C} + (p+1)A'^2 = 8\pi G_D\frac{n^2}{C^2}. \tag{3.19}$$

If we are interested in neither a coordinate singularity nor a geometry of a power law warp factor such as $A'(r) \sim r^m$ where m is a natural number, then consistency with the equations (3.18) and (3.19) allows only minimal warp geometry such as $\lim_{r\to\infty} A' =$ constant irrespective of functional form of the metric $C(r)$. Then one more set of boundary conditions is easily obtained by a simple solution of the above equations at infinity [7]:

$$A'(\infty) = \pm\frac{2\omega}{\sqrt{p+2}}, \tag{3.20}$$

$$C(\infty) = \pm\sqrt{2\pi G_D(p+2)}\,\frac{n}{\omega}. \tag{3.21}$$

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The obtained boundary conditions (3.20) and (3.21) are consistent with the remaining Einstein equation (3.5). A noteworthy character of this geometry is nonvanishing radius of the extra-dimensions represented by nonzero boundary value of the metric function C (3.21). Therefore, asymptotic region of the bulk spacetime has a finite size throat of a cylinder.

As the last test let us analyze asymptotic behavior of the scalar field, represented by small perturbation $\delta f(r)$ defined by $f(r) \approx 1 - \delta f(r)$ for sufficiently-large r. Then the scalar equation (3.4) is approximated as

$$\delta f^{''} + \left[\frac{C^{'}}{C} + (p+1)A^{'}\right]\delta f^{'} - \left(\frac{n^2}{C^2} + 2\right)\delta f \approx -\frac{n^2}{C^2}\,, \tag{3.22}$$

where smallness of the derivatives, $|\delta f^{'}| \sim |\delta f^{''}| \ll 1$, has also been forced because of smoothness of the scalar field. For the brane world of an exponentially decaying warp factor $A_-(r)$ and throat $C_-(r)$ in eqs. (2.8) and (2.9), the leading behavior of the right-hand side of eq. (3.22) forbids smallness of δf in its left-hand side, i.e., $-n^2\delta f(r)/C^2 \approx -n^2/C^2$ in eq. (3.22). It contradicts to the basic supposition of the global defect that the perturbation δf becomes small as the radius r increases at asymptotic region and reaches zero at spatial infinity. Therefore, the static regular global U(1) vortex cannot be compatible with the vacuum brane world of an exponentially decaying warp factor, that is consistent with ref. [7]. Similarly, when both $A^{'}(\infty)$ and $C(\infty)$ are nonvanishing constants as in eqs. (3.20) and (3.21), eq. (3.22) produces contradiction as follows:

$$\begin{aligned}\delta f^{''} \pm \frac{2(p+1)\omega}{\sqrt{p+2}}\delta f^{'} - \left[\frac{\omega^2}{2\pi G_D(p+2)} + 2\right]\delta f &\approx -\frac{\omega^2}{2\pi G_D(p+2)}\\ \stackrel{r\to\infty}{\Longrightarrow}\quad 0 &= -\frac{|\Lambda|}{4\pi G_D(p+1)}\,,\end{aligned} \tag{3.23}$$

where ω is replaced by eq. (2.10) in the last step of taking infinite radius. It means that the cigar-like brane world predicted in ref. [7] may not be supported by a global vortex but it is consistent with ref. [6] in the limit of vanishing cosmological constant, which involves a physical singularity believed to be harmless. In synthesis, we conclude that we could not find any regular static global U(1) vortex solution to generate a proper brane world of an exponentially decaying warp factor, yet, as far as a negative cosmological constant is turned on.

If we remind of the expansion of the scalar field and the metric functions near the origin (3.15)–(3.17), which cannot be consistent with the geometry of an exponentially decaying warp factor but can be compatible with that of an exponentially growing warp factor, it may be intriguing to look into a global vortex of which asymptote is the geometry of an exponentially decaying warp factor. From such vacuum solutions, A_+ and C_+ in eqs. (2.8) and (2.9) and A and C in eqs. (2.11) and (2.12), their leading asymptotic behavior is summarized as $A^{'}(r) \sim 2\omega/(p+2)$ and $C(r) \sim c_\infty \exp[-2\omega r/(p+2)]$. Inserting these into eq. (3.22), we have

$$\delta f^{''} + 2\omega\delta f^{'} - 2\delta f \approx 0\,. \tag{3.24}$$

This linear equation excludes power law asymptotic behavior of the scalar field, $\delta f \not\sim \mathcal{O}(1/r^m)$, but it includes an exponentially decaying asymptote, $\delta f \sim \exp[-(\omega + \sqrt{\omega^2+2})r]$, which is different from power law decaying of δf for the global vortices in both flat and curved spacetime without a cosmological constant. An unexpected exponential approach of the scalar amplitude is allowable in this coordinate system with a warp factor due to the exponential increasing of the radial circumference $C(r)$ at the asymptotic region, i.e., the topological term proportional to n^2/C^2 in the scalar equation (3.4) goes to zero in an exponential form for sufficiently-large r. Since this possibility satisfies boundary behaviors near both the origin and spatial infinity, there may exist a regular configuration to connect these two boundaries. Since n/c_0 should be a natural number in the expansion (3.15)–(3.17) near the origin, c_0 should be chosen. There remains only one free parameter f_0 in addition to three boundary values of $f(0)$, $A^{'}(0)$, $C(0)$, while we need five input parameters to obtain a solution from two second-order equations, (3.4)–(3.7), and one first-order equation, (3.6). The only adjustable parameter in eqs. (3.15)–(3.17) is $\tilde{\omega}$ in eq. (3.10), which is expressed by a difference of two fundamental scales, the D-dimensional Newton's constant G_D and a cosmological constant Λ. If there exists such a solution, then it is obtainable by a fine tuning of one of the two fundamental scales. It means that we usually obtain a solution with positive $A(r)$ for large r, which is connected to the pure AdS geometry with an exponentially growing warp factor.

Though the metric $C(r)$ increases exponentially, but ratios, $C^{'}/C$ and $C^{''}/C$, are finite and smooth at entire range. Therefore, Kretschmann invariant is finite everywhere:

$$R^{ABCD}R_{ABCD} = 4\left[2A^{'2}\left(5A^{'2} + 4A^{''} + 2\frac{C^{'2}}{C^2}\right) + \frac{C^{''2}}{C^2}\right], \tag{3.25}$$

and thereby all the spacetime points are not physically singular.

From the previous argument we have arrived at a no-go theorem that any regular static bulk global U(1) vortex cannot form a brane world of an exponentially decaying warp factor as far as the boundary condition of the scalar amplitude, $f(\infty) = 1$, is kept. One possible exit is to obtain a regular static bulk global U(1) vortex solution to support an appropriate brane world by loosening boundary conditions as minimal as we can. Boundary value of the scalar amplitude $f(\infty) = f_\infty$ is supposed to be at the range $0 < f_\infty < 1$ because all possible spacetime boundary conditions for the brane world geometry have not been compatible with $f(\infty) = 1$. Since asymptotic behavior of the scalar field f is described by a small perturbation δf given as $f(r) \approx f_\infty - \delta f(r)$, the scalar equation (3.4) is approximated as

$$\delta f^{''} + \left[\frac{C^{'}}{C} + (p+1)A^{'}\right]\delta f^{'} - \left[\frac{n^2}{C^2} + (3f_\infty^2 - 1)\right]\delta f \approx -\left[\frac{n^2}{C^2} + (f_\infty^2 - 1)\right]. \tag{3.26}$$

In the limit of infinite radius the coefficient of $\delta f^{'}$ term given by derivatives of the metric functions is known to be finite, and thereby the left-hand side of eq. (3.26) vanishes as $r \to \infty$. From the right-hand side of eq. (3.26), we read

$$f_\infty = \sqrt{1 - \frac{n^2}{C(\infty)^2}}\,. \tag{3.27}$$

If the metric function $C(r)$ approaches zero at spatial infinity as the vacuum geometry of an exponentially decaying warp factor C_- in eq. (2.9), f_∞ cannot be a real number so that the change of the scalar amplitude at spatial infinity is not enough to produce a warp geometry of the AdS vacuum. On the other hand, if $C(r)$ diverges at spatial infinity, then $f_\infty = 1$ which reduces to the original contradiction. The only choice is to let $C(\infty)$ finite but larger than the vorticity n.

Though the boundary condition of the scalar field (3.27) is not familiar to us and we have no clear physicswise motivation to accept such awkward soliton configuration yet, we try first to confirm the existence of such configuration. A possible power series solution is assumed as follows:

$$f(r) \approx f_\infty - \frac{f_u}{r^u} + \cdots\,, \tag{3.28}$$

$$A'(r) \approx a_\infty + \frac{a_v}{r^v} + \cdots\,, \tag{3.29}$$

$$C(r) \approx c_\infty + \frac{c_w}{r^w} + \cdots\,, \tag{3.30}$$

where boundary values of the scalar field f_∞ and the metric functions a_∞, c_∞ do not vanish. Since only algebraic terms in the equations (3.4), (3.6) and (3.7) are needed for determining the leading terms so that a possible set of boundary conditions at spatial infinity are given as

$$f_\infty = \sqrt{1-\xi^2} < 1\,, \qquad a_\infty = -\frac{2}{p+1}\sqrt{\frac{p|\Lambda|}{2p+1}}\,, \qquad c_\infty = \frac{n}{\xi}\,, \tag{3.31}$$

where $\xi = [|\Lambda|/2\pi(2p+1)G_D]^{1/4}$ and $2\pi(p+1)G_D > |\Lambda|$. Since $C(\infty) = c_\infty$ is proportional to n, there is no constraint to the vorticity n from the boundary condition (3.27). From the leading behavior at asymptotic region, a cigar-like geometry after a rescaling of x^μ and ds is summarized by a metric with an exponentially decaying warp factor:

$$ds^2 = \exp\left(-\frac{4}{p+1}\sqrt{\frac{p|\Lambda|}{2p+1}}\,r\right)dx^\mu dx_\mu - dr^2 - n^2\sqrt{\frac{2\pi(2p+1)G_D}{|\Lambda|}}d\theta^2\,. \tag{3.32}$$

Systematic expansion up to the next order allows a unique series solution:

$$f(r) \approx \sqrt{1-\xi^2}\left(1 + \frac{p}{5p+2}\frac{\xi^2\eta}{1-\xi^2}\frac{1}{r} + \cdots\right), \tag{3.33}$$

$$A'(r) \approx -\frac{2}{p+1}\sqrt{\frac{p|\Lambda|}{2p+1}}\left(1 + \frac{\eta}{5p+2}\frac{1}{r} + \cdots\right), \tag{3.34}$$

$$C(r) \approx \frac{n}{\xi}\left(1 + \frac{p\eta}{5p+2}\frac{1}{r} + \cdots\right), \tag{3.35}$$

where

$$\eta = \frac{p+1}{2}\sqrt{\frac{2p+1}{p|\Lambda|}}\left[1 + \frac{4p|\Lambda|}{(p+1)(2p+1)(1-\xi^2)}\right]. \tag{3.36}$$

Note that all the expansion parameters in eqs. (3.28)–(3.30) are determined by the Newton's constant G_D, a cosmological constant Λ, and the vorticity n. It means that such configuration is achieved by the aforementioned fine tuning. This phenomenon was already expected from the behavior of the expansion near the origin (3.15)–(3.17). In the next section, a clear explanation on this fine tuning will be given after a coordinate transformation.

If we extend the form of metric functions to $A(r) \propto r^s$ $(s > 1)$ and $C(r) \propto \exp\left(\tilde{c}r^{t_1}\right)$ or $C(r) \propto r^{t_2}$, we encounter easily a mismatch in the expansion of series solution. Furthermore, change of the functional behavior $f(r)$ from a power series to an exponential form $f(r) \sim f_\infty - f_e \exp(-\hat{c}r)$ does not help for the generation of smooth solution anymore. Therefore, the obtained solution (3.33)–(3.35) must be unique regular static global U(1) vortex solution with a brane world geometry of an exponentially decaying warp factor up to now by massaging the boundary condition (3.27) at a physicswisely-acceptable and minimal extent.

4. Black brane world and short scalar hair

For the global p-brane obtained in the previous section, the scalar amplitude f did not reach the vacuum expectation value even at infinite r as in eq. (3.31). It means that there is a contribution from the scalar potential to nonvanishing effective cosmological constant at $r = \infty$

$$\lambda v^2 \left[\Lambda + 2\pi G_D ({f_\infty}^2 - 1)^2\right] = -\frac{2p}{1+2p}\lambda v^2 |\Lambda| \,. \tag{4.1}$$

In addition, boundary values of the metric functions (3.31) are also different from eqs. (3.20)–(3.21). Asymptotic geometry depicts qualitatively the same cigar-like (or cylindrical) geometry and includes an exponentially decaying warp factor in front of the p-brane coordinates except for a few quantitative changes: the radius of the cigar becomes $C(\infty) = n/[|\Lambda|/2\pi(2p+1)G_D]^{1/4}$ and the decaying in the exponential warp factor $A'(\infty) = -\frac{2}{p+1}\sqrt{\frac{p|\Lambda|}{2p+1}}$ at spatial infinity. Through the boundary condition of the scalar field at infinity of the extra-dimensions was drastically changed, the obtained asymptotic geometry is qualitatively the same as that under the assumption $f(\infty) = 1$ [7]–[9] and [11, 16, 18].

Now we have to address a question what kind of physical interpretation is applicable to this awkward boundary condition of the scalar field f. In the curved spacetime, a natural conjecture may be given as follows. The bulk spacetime with a warp factor, obtained in this coordinate system (2.2), does not describe entire spacetime but a patch of it so that a point of spatial infinity in the extra-dimensions corresponds to a mid-point of the entire geometry which is describable in another coordinate system. An appropriate coordinate system may be the aforementioned Poincaré coordinates (2.15), and, in this metric, the equation of the scalar field f (3.4) is written as

$$B\frac{d^2 f}{dR^2} + \left[\frac{B}{R} + \left(1+\frac{p}{2}\right)\frac{dB}{dR} + (1+p)B\frac{d\Phi}{dR}\right]\frac{df}{dR} - \left[\frac{n^2}{R^2} + (f^2-1)\right]f = 0\,. \tag{4.2}$$

As shown in ref. [11, 16, 19], the metric functions involve a horizon at $R = R_{\rm H}$ and the near horizon geometry is shown to behave

$$B(R) \sim (R - R_{\rm H})^2 \qquad \text{and} \qquad \frac{d\Phi}{dR} \sim \frac{1}{R - R_{\rm H}}\,. \tag{4.3}$$

Therefore, presumably, the location of the horizon at $R_{\rm H}$ in the Poincaré coordinates (2.15) corresponds to the infinity ($r \to \infty$) of the metric with a warp factor (2.2), and the scalar equation in both metrics, eq. (3.4) and eq. (4.2), reduce exactly to the following algebraic equations at their boundaries:

$$\left[\frac{n^2}{C(r)^2} + (f(r)^2 - 1)\right]\Bigg|_{r\to\infty} = 0 \Longleftrightarrow \left[\frac{n^2}{R^2} + (f(R)^2 - 1)\right]\Bigg|_{R=R_{\rm H}} = 0\,. \tag{4.4}$$

Though our argument was based on a specific form of ϕ^4 scalar potential, every argument can also be applied to any scalar potential $V(f)$ once eq. (4.4) is replaced by

$$\left[\frac{n^2}{C(r)^2} + \frac{1}{f}\frac{dV}{df}\right]\Bigg|_{r\to\infty} = 0\,, \tag{4.5}$$

and it contains a solution $f(\infty)$ smaller than the vacuum expectation value.

It implies that the finite boundary value of C at infinity should coincide with the finite radius of the horizon $R_{\rm H}$ proportional to the vorticity n

$$R_{\rm H} = \lim_{r\to\infty} C(r) = \frac{n}{\xi}\,, \tag{4.6}$$

$$f(R_{\rm H}) = \lim_{r\to\infty} f(r) = \sqrt{1-\xi^2}\,, \tag{4.7}$$

where ξ is given around eq. (3.31). Furthermore, at asymptotic region of r, performing a coordinate transformation from eq. (3.32)

$$e^{2a_\infty r} = (R_{\rm H} - \rho)^2 \tag{4.8}$$

with a rescaling, $d\tilde{s} = -a_\infty ds$ and $d\tilde{x}^\mu = -a_\infty dx^\mu$, we obtain the metric of an extremal black p-brane

$$d\tilde{s}^2 = (R_{\rm H} - \rho)^2 d\tilde{x}^\mu d\tilde{x}_\mu - \frac{d\rho^2}{(R_{\rm H} - \rho)^2} - R_{\rm H}^2 d(-a_\infty\theta)^2\,, \tag{4.9}$$

where a_∞ is also given in eq. (3.31). Since $-a_\infty \neq 1$ in usual cases, a deficit angle is identified at the generated horizon due to a long range topological term of scalar phase. This phenomenon is consistent with the short scalar hair of extremal charged BTZ black hole [21], since it is nothing but a black 0-brane in our context.

We give a rough explanation on the formation of extremal black p-brane with degenerated horizon in the scheme of Hodge duality on the p-brane [22] as has been done in $p = 0$ case [21]. In the $(p+3)$-dimensional bulk, the U(1) current J_A is defined by and rewritten in terms of its dual $(p+2)$-form field strength tensor $H^{A_1A_2\cdots A_{p+2}}$ as

$$J_A = v^2 f^2 \partial_A \Omega \sim \sqrt{-g}\,\epsilon_{ABC_0C_1\cdots C_p} H^{BC_0C_1\cdots C_p}\,. \tag{4.10}$$

Then, the corresponding Lagrange densities satisfy the following relation up to a multiplicative constant

$$v^2 f^2 g^{AB}\partial_A\Omega\partial_B\Omega \sim \frac{1}{v^2 f^2} g^{A_1B_1}g^{A_2B_2}\cdots g^{A_{p+2}B_{p+2}} H_{A_1A_2\cdots A_{p+2}} H_{B_1B_2\cdots B_{p+2}}\,. \tag{4.11}$$

The scalar phase Ω was given by $\Omega = n\theta$ for global defects in the extra-dimensions, so the single electric component survives as follows $H_{t12\cdots pr} \sim C(r)^{-1}e^{(p+1)a_\infty r}v^2 f^2 n$. In the field strength tensor, p transverse coordinates do not play any role except for introducing warp factors. Substituting it into the Lagrange density (4.11), we see that at asymptotic region quadratic term of the dual $(p+2)$-form field strength tensor remains to be a constant term independent of the warp factor

$$\frac{1}{v^2 f^2} g^{rr}g^{tt}g^{11}\cdots g^{pp}H^2_{t12\cdots pr} \sim v^2 f^2 g^{\theta\theta}(\partial_\theta\Omega)^2 = v^2 f^2\frac{n^2}{C^2} \stackrel{r\to\infty}{\longrightarrow} v^2 f_\infty^2 \frac{n^2}{c_\infty^2}\,, \tag{4.12}$$

and then structure of the Hodge duality is almost the same as that of any p. At the origin of $r = 0$, $1/f^2$-factor in eq. (4.11) plays an important role to make the site of p-brane regular. So there is a significant difference between the Hodge-dual field strength with the scalar amplitude f and a pure $(p+2)$-form field strength tensor since we should live on that singular p-brane in the latter case.

Let us examine the behavior of the scalar field and the metric functions near the origin and show that those in both coordinate systems, eq. (2.2) and eq. (2.15), behave consistently each other up to the leading approximation. For the scalar field, it should vanish at the origin such that $f(r = 0) = f(R = 0) = 0$ and its slope near the origin is the same as shown in eq. (3.15) and ref. [11]. Substituting the expansion of the metric functions (3.16)–(3.17) into eq. (2.2), we obtain

$$ds^2 \approx \left(1 - \frac{2\pi G_D - |\Lambda|}{p+1}r^2\right)dx^\mu dx_\mu - dr^2 - r^2 d\theta^2\,. \tag{4.13}$$

A coordinate transformation from eq. (4.13) to the Poincaré coordinate system (2.15) results in the same metric obtained by the power series expansion in the Poincaré coordinate system. Since all the field and the metric functions are smooth between the origin and spatial infinity where the boundary values coincide, it leads to a conclusion that the obtained cigar-like bulk geometry with an exponentially decaying warp factor parametrized by r-coordinate ($0 \le r \le \infty$) coincides in fact with a patch in the coordinate system of R ($0 \le R \le R_{\rm H}$), which is interior region bounded by the extremal black hole horizon as was the case of the vacuum solution in the previous section.

A few comments on comparison of the geometric properties of two bulk spacetimes are in order. First, infinite radial distance $\int_0^\infty dr$ in this brane world corresponds to infinite proper distance to the horizon of the extremal black hole, $\int_0^{R_{\rm H}} dR/\sqrt{B(R)} \sim \lim\limits_{R\to R_{\rm H}} \ln(R_{\rm H} - R) \sim \infty$. Second, finiteness of the bulk volume of the brane world divided by the spacetime volume of the p-brane such as

$$\frac{\int d^{p+3}x\sqrt{|g_{p+3}|}}{\int d^{p+1}x} = 2\pi\int_0^\infty dr\, e^{(p+1)A(r)}C(r) \sim \text{finite} \tag{4.14}$$

can be interpreted as the finite volume inside the horizon divided by the spacetime volume of the p-brane:

$$\frac{\int d^{p+3}x\sqrt{|g_{p+3}|}}{\int d^{p+1}x} = 2\pi\int_0^{R_{\rm H}} dR\, Re^{(p+1)\Phi}B^{p/2} \sim \text{finite}\,. \tag{4.15}$$

Since $A(r\to\infty)$ is independent of the vorticity n and $C(r\to\infty)$ is proportional to n as given in eq. (3.31), the bulk volume of the brane world divided by the spacetime volume of the p-brane (4.14) (or equivalently, eq. (4.15)) can be very small when n is small and very large when n becomes a huge number. Noticing that the only variable parameter in the algebraic equation (4.4) is topological charge n for a given scale, the very process may provide a mechanism to interpolate the RS model I of small extra-dimensions to the ADD model of large extra-dimensions by quantity of the topological charge n. However, we have no physical reason to fix the vorticity n in our present scheme, so choice of an n is also a fine-tuning though it seems mild.

The property that the scalar field can approach the vacuum value $f=1$ at spatial infinity is available only when the circumference proportional to C/n goes to infinity as has been read in eq. (4.4). Since $C(\infty)/n = 1/[|\Lambda|/2\pi(2p+1)G_D]^{1/4}$, such a limit can be achieved by the limit of vanishing $|\Lambda|/G_D$. After a rescaling back to the original variables by using eq. (3.3), we have $|\Lambda|/G_D \to \lambda|\Lambda|M_*^{p+1}$ which implies either $\lambda\to 0$ or $|\Lambda|M_*^{p+1}\to 0$ in order to have $\lim_{r\to\infty} f(r)=1$. Here $\lambda\to 0$ seems an unwanted situation. It means that the extremal black hole becomes free from the short hair of scalar amplitude, but it seems not so interesting in the context of black hole physics since the radius of the horizon moves far away to the infinity, $R_{\rm H}\to\infty$, by the right-hand side of eq. (4.4). Furthermore, the limit of vanishing cosmological constant heads for appearance a physical singularity at infinity [6]. Let us emphasize again that the limit of $\lim_{r\to\infty} f=1$ cannot be taken as far as $|\Lambda|/G_D$ is finite even if the topological charge n diverges, and thereby the existence of a short scalar hair is unavoidable in the case of global U(1) defect without an unnatural fine tuning.

In relation with an extension to arbitrary $N(\geq 3)$ extra-dimensions with O(N) linear σ-model and their higher-dimensional analogues [8, 11, 9, 14] probably develop a short scalar hair, since the topological term proportional to n^2/C^2 both in eq. (3.4) and eq. (3.7) exists even in higher extra-dimensions with unit topological charge ($n=1$). The previous analysis of the black brane world based on half σ-lumps in O($N+1$) nonlinear σ-model [23, 19] provides a positive hint on a short scalar hair, of which results were independent of the extra-dimensions N.

5. Discussion

In this paper we considered a bulk complex scalar field in a warp geometry and the global vortices in rotationally symmetric two extra-dimensions, which form a flat thick p-brane. The corresponding brane world is identified with the interior of an extremal black brane world bounded by a horizon. An unavoidable short hair of the scalar amplitude on the horizon is translated as its boundary value at infinity in the metric with an exponentially decaying warp factor. Despite this short scalar hair, the obtained warp geometry is qualitatively the same as that in ref. [7].

In section 2, we obtained general static brane world solutions with two extra-dimensions, but it is intriguing to extend our results including general time-dependent solutions as in the case of one extra-dimension [24] or that in arbitrary extra dimensions [25, 26]. These questions are also applied to the black p-brane world solutions formed by global vortices in the two extra-dimensions. Particularly, fate of the short scalar hair should be asked when time dependence is taken into account [18].

In our approach we have assumed a bulk complex scalar field in (1+5)-dimensions, and considered a global vortex solution in two extra-dimensions to interpret our world as 3-brane of codimension-two. Then gapless Goldstone modes can be generated outside the horizon in the broken phase and, despite of the existence of this horizon, those energetic modes can reach the 3-brane of our world, where they may threaten conservation of energy. This implies a limitation of this toy model based on local field theory, and it should be improved by the model from string theory. A natural way is to understand our (1+5)-dimensional bulk with a complex scalar field as a space-filling D5-$\bar{\text{D}}$5 pair. The decent relations among D-branes in string theory dictate the decay of D5-$\bar{\text{D}}$5 pair to codimension-two brane and its dynamics is described by condensation of a complex tachyon [27]. After the D5-$\bar{\text{D}}$5 pair decays into codimension-two brane, all the open string modes including Goldstone modes should disappear in the extra-dimensions. In such sense, an appropriate effective field theory model derived from string theory is chosen and should be tackled. Though the effective field theory of tachyon is usually depicted by Born-Infeld type nonlocal action with runaway potential [28], tachyon vortices may share similar characters with global vortices in various features.

Acknowledgments

This work is supported by Korea Research Foundation Grants KRF-2001-015-DP0082 and is the result of research activities (Astrophysical Research Center for the Structure and Evolution of the Cosmos (ARCSEC)) supported by Korea Science & Engineering Foundation.

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